COSMOLOGICAL IMPLICATIONS OF A VIABLE NON-ANALYTICAL *f(R)*-GRAVITY MODEL

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We show how power-law corrections to the Einstein-Hilbert action yield a viable extended theory of gravity, passing the Solar-System tests, provided that the power-law exponent n is strictly comprised between 2 and 3. We implement this paradigm on a cosmological setting outlining how the main phases of the Universe thermal history are properly reproduced. As a result, we find two distinct constraints on the characteristic length scale of the model, *i.e.*, a lower bound from the Solar-System test and an upper bound found by requiring the existence of the matter-dominated phase of the Universe evolution.

We also show how the extended framework can accommodate the existence of an early de Sitter phase. Within the allowed range of characteristic length scales, the relation between the expansion rate H_I and the energy scale M of inflation is modified, yielding a value of H_I several orders of magnitude smaller than the one found in the standard picture, *i.e.*, $\overline{H}_I \sim M^2/m_{\rm pl}$. The observational implication of this fact is that, quite generally, a tiny value of the tensor-to-scalar ratio r is expected in the extended framework, that will go undetected even by future missions focused on cosmic microwave background polarization, like CMBPol. The suppression of primordial tensor modes also implies that the inflationary scale can be made arbitrarily close to the Planck one without spoiling the current limits on r.

Finally, considering the same modified action, an analysis of the propagation of gravitational waves on a Robertson-Walker background is addressed. We find that, in the allowed parameter range, the $f(R)$ correction does not significantly affect the standard evolution.

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I. BASIC STATEMENTS

Several issues in cosmology and quantum field theory seem to suggest that Einstein's General Relativity (GR) should be extended in order to cure shortcomings emerging from observations and self-consistent unification theories. In early-time cosmology, the presence of the big bang singularity, flatness and horizon problems [1] led to the conclusion that Standard Cosmological Model [2] is inadequate to describe the Universe at extreme regimes. On the other hand, GR is a classical theory that is not expected to work as a fundamental theory, when a full quantum description of spacetime is required. Due to these facts and, first of all, to the lack of a self-consistent Quantum Gravity theory, alternative theories of gravity have been pursued in order to find, at least, a consistent semi-classical scheme where GR and its successes could be recovered. Extended Theories of Gravity (in particular $f(R)$ gravity) represent a fruitful approach and have become a sort of paradigm in the study of gravitational interaction based on corrections and extensions of the Einstein scheme [3–5]. Beside fundamental physics motivations, these theories have acquired a huge interest in cosmology since they "naturally" exhibit inflationary behaviors able to overcome the shortcomings of the Standard Cosmological Model and the associated schemes are, in many cases, able to match the observations [6–10].

The appeal of Extended Theories of Gravity also resides in the fact that they could allow to explain the gravitational anomalies, usually interpreted in terms of "dark" components of the cosmological fluids, without invoking exotic physics, *i.e.*, physics beyond the standard model of particle physics [11]. In fact, GR has been successfully directly probed only in the weak-field limit (*e.g.*, Solar-System experiments) and also in this case there is room for the implementation of alternative theories of gravity which are not at all ruled out, as discussed in many recent studies [12, 13]. In particular, it is possible to show that several $f(R)$ models can satisfy both cosmological and Solar-System tests [14, 15] and they can be constrained as the scalar-tensor theories. Moreover, such theories give rise to new effects able to explain the so-called Pioneer anomaly (see for example [16] and references therein). It is important to remark that any modification of the Einstein-Hilbert (EH) Lagrangian is reflected onto a deformed gravitational field dynamics at any length scale investigated or observed. Thus, the success of $f(R)$ gravity in the solution of a specific problem has to match consistency with observation in different length scales [10]. In this respect, we emphasize that the small value of the present curvature of the Universe [17] leads us to believe that, independently of its specific functional form, the $f(R)$ term must be regarded as a lower-order expansion in the Ricci scalar. On the other hand, it is easily understood that the peculiarities of such an expansion will be extremely sensitive to the morphology of the deformed Lagrangian.

The most immediate generalization of the EH action is to deal with a function of the Ricci scalar R analytical in the point $R = 0$, so that its Taylor expansion holds [13]. This approach is equivalent to deal with a polynomial form of $f(R)$ [18, 19], whose free parameters are available to fit the observed phenomena on different sectors of investigation. Despite the appealing profile of such a choice, it is extremely important to observe that it is not the most general case, since real (non-integer) exponents of R are, in principle, on the same footing as the simplest case [20].

In this paper, we focus on such an open issue and we will develop a modified theory of the form $f(R) = R + qR^n$, where q and n are two free parameters to be constrained at physical level [8, 10]. In [12], it is shown how the details of the model lead to deal with rational non-integer numbers for n and to restrict it in the most appropriate interval for physical interpretation at low curvature, $2 < n < 3$. In fact, by analyzing the weak-field limit of our modified Lagrangian and retaining only those non-integer powers in R which represent the dominant effects after the linear approximation, the choice $2 < n < 3$ allows us to distinguish the dominant non-Einsteinian terms from the non-linear ones of GR. The explicit solution of the system derived in [12] is here re-analyzed confirming the existence of a lower bound for the characteristic length scale of the model (defined by means of the parameter q) and of two other typical lengths which guarantee that our $f(R)$ model fulfills the Solar-System constraints.

The cosmological study of the resulting modified Friedmann-Lemaître-Robertson-Walker (FLRW) dynamics is also addressed as the central theme of this paper. The fundamental equation for the scale factor is derived for a perfect fluid matter source and the main phases of the Universe evolution, *i.e.*, the radiation- and matter-dominated era, are separately discussed. By assuming a generic power-law (in time, t) behavior of the scale factor, we show how the two phases are asymptotically properly reproduced. In particular, in the asymptotic limit toward the singularity (taken in $t = 0$), the radiation-dominated scale factor expansion $\sim t^{1/2}$ is recovered for all values of the FLRW spatial curvature and a phase of power-law inflation ($\sim t^{n/2}$) can be generated for vanishing curvature. On the other hand, in the limit of large t, the matter-dominated law for the scale factor $\sim t^{2/3}$ is reproduced neglecting the spatial curvature.

The main result of the cosmological analysis is the existence of an upper bound of ~ 70 pc (weakly dependent on the value of n for $2 < n < 3$) for the characteristic length scale of the model which guarantees the correct matter-dominated Universe evolution. Combining this cosmological bound with the Solar-System constraints, we define the allowed values of the parameter q in our $f(R)$ scheme, thus providing a restricted region of the parameter space in which the model can be implemented both in the weak-field limit and in the cosmological framework.

After the analysis of the power-law Universe evolution, a study of the existence of the standard (exponential)

inflationary behavior in our generalized framework is also provided. In particular, we show how an exponential early evolution of the Universe is still present in the modified scheme; however, the existence of a minimum length scale for the model, as imposed by the Solar-System tests, implies that the expansion rate H_I is much smaller than the standard one, *i.e.*, $\overline{H}_I \sim M^2/m_{\rm pl}$, by several orders of magnitude. We then discuss the observational implications of this modified dynamics, underlying how the production of tensor perturbations is much less efficient than usual. This allows to have the inflationary scale M arbitrarily close to the Planck one without spoiling the current limits on the tensor-to-scalar ratio r. More generally, we express r as a function of the parameters of the model and of M and we find that a tiny value of the tensor-to-scalar ratio is expected in the extended framework. In particular, it is very difficult to obtain a value of $r \gtrsim 10^{-80}$ unless the exponent n is fine tuned to be very close to 2. Such a small value of r will be impossible to detect even for future space-based missions focused on cosmic microwave background (CMB) polarization, like CMBPol.

Finally, we shortly focus on the study of the gravitational wave (GW) propagation on a Robertson-Walker (RW) background. The GW equation is derived, using the standard conformal formalism, in the transverse-traceless gauge, firstly without specifying the form of $f(R)$. This way, we obtain a general propagation equation for GWs in the case of a flat RW model. We then assume a power-law behavior of the scale factor and our non-analytical form of $f(R)$. We find that the GW propagation is unchanged during the radiation-dominated era, but can be different from standard GR, in principle, during the matter-dominated era. However, the existence of a maximum length scale implies that the $f(R)$ corrective term can be always neglected during this epoch, and thus does not yield any observational consequence.

The paper is organized as follows. In Section 2, after a brief review of the field equations of $f(R)$ gravity in the metric approach, we introduce and discuss our viable $f(R)$ model. In Section 3, we recall and extend the implementation of the Solar-System test to our model [12] and derive the lower bound to the length scale associated to the $f(R)$ corrections to GR. In Section 4, we carry a study of the resulting modified FLRW dynamics and the inflationary paradigm. As expected, this scenario gives us a rather stringent upper constraint to the characteristic length scale of the model. In Section 5, we discuss the propagation of cosmological GWs in the presence of $f(R)$ gravity.

Notations – The signature is set as $[-,+,+,+]$; Greek indices run form 0 to 3; Latin indices run from 1 to 3; $(...)'$ indicates the derivative with respect to R; ∇_μ or $(...)_;$ denotes the covariant derivative; The dot $(...)^{\cdot}$ is the time derivative; $(...)_,$ indicates ordinary differentiation; $\Box \equiv \nabla^\mu \nabla_\mu$; We use natural units $c = \hbar = 1$ and we define $\chi \equiv 8\pi G$, G being the Newton constant; $m_{\rm pl} \equiv 1/\sqrt{G} \simeq 10^{19}$ GeV is the Planck mass.

II. A NON-ANALYTICAL POWER-LAW F(R)-MODEL

In this paper, we consider the following modified gravitational action in the so-called *Jordan frame*

$$\mathcal{S} = \tfrac{1}{2\chi} \int d^4x \sqrt{-g}\, f(R)\,, \qquad\qquad f(R) = R + qR^n\,, \tag{1}$$

where n is a non-integer dimensionless parameter and $q < 0$ has dimensions $[L]^{2n-2}$. Such a form of $f(R)$ gives the following constraints for n: if $R > 0$, all n-values are allowed; if $R < 0$, the condition $n = \ell/(2m+1)$ must hold (where, here and in the following, m and ℓ denote positive integers). It is straightforward to verify that S in Eq.(1) is non-analytical in $R = 0$ for non-integer and rational values of n, *i.e.*, it does not admit Taylor expansion near $R = 0$.

Let us now define the *characteristic length scale* of our model as

$$L_q(n) \equiv |q|^{1/(2n-2)}\,, \tag{2}$$

while variations of the total action $\mathcal{S}_{tot} = \mathcal{S} + \mathcal{S}_M$ (where $\mathcal{S}_M$ denotes the matter term) with respect to the metric give, after manipulations and modulo surface terms:

$$f'R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} f - \nabla_\mu \nabla_\nu\, f' + g_{\mu\nu} \Box\, f' = \chi\, T_{\mu\nu}\,, \tag{3}$$

where $T_{\mu\nu}$ is the Energy-Momentum Tensor (EMT).

We can gain further information on the value of n by analyzing the conditions that allow for a consistent weak-field stationary limit. Having in mind to investigate the weak field limit of our theory to obtain predictions at Solar-System scales, we can decompose the corresponding metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a small (for our case, static) perturbation of the Minkowskian metric $\eta_{\mu\nu}$. In this limit, the vacuum Einstein equations read

$$R_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} R - nq(R^{n-1})_{;\mu;\nu} + nq\eta_{\mu\nu}\Box R^{n-1} = 0\,, \qquad\qquad R = 3nq\Box R^{n-1}\,. \tag{5}$$

The structure of such field equations leads us to focus our attention on the restricted region of the parameter space $2 < n < 3$. This choice is enforced by the fulfillment of the conditions by which all other terms are negligible with respect to the linear and the lowest-order non-Einsteinian ones.

III. VIABILITY OF THE MODEL: THE SOLAR-SYSTEM TESTS

From the analysis of the weak-field limit in the Jordan frame, *i.e.*, Eqs.(5), we learn the possibility to find a post-Newtonian solution by solving Eqs.(5) up to the next-to-leading order in h, *i.e.*, up to $\mathcal{O}(h^{n-1})$, and neglecting the $\mathcal{O}(h^2)$ contribution only for the cases $2<n<3$. These considerations motivate the choice we claimed above concerning the restriction of the parameter n. The most general spherically-symmetric line element in the weak-field limit writes as

$$-\,ds^2=(1+\Phi)dt^2-(1-\Psi)dr^2-r^2d\Omega^2\,, \tag{6}$$

where Φ and Ψ are the two generalized gravitational potentials and $d\Omega^2$ is the solid angle element. Within this framework, the modified Einstein equations (5) can be integrated [12] to obtain the following solutions:

$$R=Ar^{\frac{2}{n-2}}\,,\qquad A=\left[-\frac{6nq(3n-4)(n-1)}{(n-2)^2}\right]^{\frac{1}{2-n}}\,, \tag{7a}$$

$$\Phi=\sigma+\frac{\delta}{r}+\Phi_n\left(\frac{r}{L_q}\right)^{2\,\frac{n-1}{n-2}}\,,\qquad \Phi_n\equiv\left[-\frac{6n(3n-4)(n-1)}{(n-2)^2}\right]^{\frac{1}{2-n}}\frac{(n-2)^2}{6(3n-4)(n-1)}\,, \tag{7b}$$

$$\Psi=\frac{\delta}{r}+\Psi_n\left(\frac{r}{L_q}\right)^{2\,\frac{n-1}{n-2}}\,,\qquad \Psi_n\equiv\left[-\frac{6n(3n-4)(n-1)}{(n-2)^2}\right]^{\frac{1}{2-n}}\frac{(n-2)}{3(3n-4)}\,, \tag{7c}$$

where the integration constant δ has the dimensions $[L]$ and the dimensionless integration constant σ can be set equal to zero without loss of generality. The integration constant A has the dimensions $[L]^{(2n-2)/(2-n)}$, and Φ_n and Ψ_n are dimensionless, accordingly. Moreover, one can check that Φ_n and Ψ_n are well-defined only in the case $n=(2m+1)/\ell$ while we get $A>0$ since we assume $q<0$. In agreement to the geodesic motion as expanded in the weak field limit, the integration constant δ results equal to $\delta=-r^S$, where $r^S=2GM$ is the *Schwarzschild radius* of a central object of mass M.

The most suitable arena to evaluate the reliability and the validity range of the weak-field solution (7) is the Solar System [8, 10]. To this end, we can specify Eq.(7b) and Eq.(7c) for the typical length scales involved in the problem and we split Φ and Ψ into two terms, the Newtonian part and a modification, *i.e.*,

$$\Phi\equiv\Phi_N+\Phi_M\equiv-r^S_\odot/r+\Phi_n(r/L_q)^{2(n-1)/(n-2)}\,, \tag{8a}$$

$$\Psi\equiv\Psi_N+\Psi_M\equiv-r^S_\odot/r+\Psi_n(r/L_q)^{2(n-1)/(n-2)}\,, \tag{8b}$$

here, the integration constant δ of Eqs.(7b)-(7c) is $\delta=-r^S_\odot\equiv-2GM_\odot$ ($M_\odot$ being the Solar mass). While the weak-field approximation of the Schwarzschild metric is valid within the range $r^S_\odot\ll r<\infty$ because it is asymptotically flat, the modification terms have the peculiar feature to diverge for $r\to\infty$. It is therefore necessary to establish a validity range, *i.e.*, $r_{Min}\ll r\ll r_{Max}$, related to n and L_q, where this solution is physically predictive [21].

Since we aim to provide a physical picture at least of the planetary region of the Solar System, we are led to require that Φ_M and Ψ_M remain small perturbations with respect to Φ_N and Ψ_N, so that it is easy to recognize the absence of a minimal radius except for the condition $r\gg r^S_\odot$. One can now define a typical distance $L^*_\odot$ corresponding to the distance where the Newtonian and post-Newtonian terms are of the same order. Thus, for $r^S_\odot\ll r\ll L^*_\odot$, the system obeys thus Newtonian physics and experiences the post-Newtonian term as a correction. Another maximum distance $L^{**}_\odot$ can be futhermore considered, following from the request that the weak-field expansion (6) should hold, regardless of the ratios Φ_M/Φ_N and Ψ_M/Ψ_N. $L^*_\odot$ and $L^{**}_\odot$ result to be defined as functions of n and L_q and they write as [12, 22]

$$L^*_\odot\sim|r^S_\odot/\Phi_n|^{\frac{n-2}{3n-4}}L_q^{\frac{2n-2}{3n-4}}\,,\qquad L^{**}_\odot\sim L_q/|\Phi_n|^{\frac{n-2}{2n-2}}\,, \tag{9}$$

and, for the validity of our scheme, $L^*_\odot\gg r^S_\odot$ must hold, *i.e.*, $L_q\gg r^S_\odot|\Phi_n|^{(n-2)/(2n-2)}$.

Neglecting the lower-order effects concerning the eccentricity of the planetary orbit, we can deal with the simple model of a planet moving on circular orbit around the Sun with an orbital period T given by $T=2\pi(r/a)^{1/2}$, where $a=(d\Phi/dr)/2$ is the centripetal acceleration. We can now compare the correction to the Keplerian period $T_K=2\pi r^{3/2}(GM_\odot)^{-1/2}$, with the experimental data of the period T_{exp} and its uncertainty δT_{exp}. We then impose that the correction arising from the Φ_M term in Eq.(7b) is smaller than the experimental uncertainty, *i.e.*, $\delta T_{exp}/T_{exp}\geqslant|T_K-T|/T_K$. Specifying now our analysis for the case of the Earth [8], *i.e.*, $T_{exp}\simeq365.2563$ days and $\delta T_{exp}\simeq5.0\cdot10^{-10}$ days, we can get a lower bound $L_q>L_q^{Min}$ for the characteristic length scale [22], as function of n. For

our model, from Eqs.(7b), it writes

$$L_q^{Min}(n) = \left[1.3689 \times 10^{12}\,\frac{|\Phi_n|}{r_\odot^S}\,\frac{n-1}{n-2}\,r_P^{\frac{3n-4}{n-2}}\right]^{\frac{n-2}{2n-2}}\,, \tag{10}$$

where $r_P \simeq 4.8482\times10^{-6}$ pc is the mean orbital distance of Earth from the Sun. The minimum length L_q^{Min} increases monotonically for $2 < n < 3$; it tends to a finite value $L_q^{Min} \simeq 0.05$ pc when $n \to 3$, while for $n \to 2$ it tends to 0 with the asymptotic behaviour $L_q^{Min} \approx 10^{-6}(n-2)$. For a typical value $n \simeq 2.66$, one gets $L_q^{Min} \sim 4\times10^{-3}$ pc. We remark that L_q^{Min}, by virtue of Eq.(7b), is defined only for $n = (2m+1)/\ell$.

Our analysis clarifies how the predictions of the corresponding equations for the weak-field limit appear viable in view of the constraints arising from the Solar-System physics. Indeed, the lower bound for L_q does not represent a serious shortcoming of the model, as we are going to discuss in Section IV C.

IV. COSMOLOGICAL IMPLEMENTATION OF THE MODEL

In order to study how our *f(R)* model affects the cosmological evolution, we start from the modified gravitational action (1) and we assume the standard RW line element in the synchronous reference system, *i.e.*,

$$-ds^2 = dt^2 - a(t)^2\left[\,dr^2/(1-Kr^2) + r^2 d\Omega^2\,\right]\,, \tag{11}$$

where $a(t)$ is the scale factor and K the spatial curvature constant. With this form of the metric, the 00-component of Eq.(3) results, using the Bianchi identity, to be the only independent one and writes as

$$f' R_{00} + \tfrac{1}{2}\,f - 3(\dot a/a)\,f''\,\dot R = -\chi\, T_{00}\,. \tag{12}$$

We assume as matter source a perfect-fluid EMT, *i.e.*, $T_{\mu\nu} = (p+\rho)\,u_\mu u_\nu + p g_{\mu\nu}$, in a comoving reference system (thus $T_{00} = \rho$), where p is the thermostatic pressure, ρ the energy density and u_μ denotes the 4-velocity. Assuming the *equation of state* (EoS) $p = w\rho$, the $\mu = 0$ component of the conservation law $T^\nu_{\mu;\,\nu} = 0$ gives the following expression for the energy density: $\rho = \rho_0\,[a/a_0]^{-3(1+w)}$.

Using now $f = R + qR^n$ with $q < 0$, we are able to explicitly write Eq.(12):

$$\begin{aligned}
&2\,\tilde\chi\, a^{1-3w} \;+\; 6^n\, n\, q\, a^{5-2n}\,\ddot a\,(-K - \dot a^2 - a\,\ddot a)^{n-1} \,+\\
&+ a^2\left[\,-6K - 6\,\dot a^2 + 6^n\, q\, a^{2(1-n)}\,(-K - \dot a^2 - a\,\ddot a)^n\right] +\\
&+ 6^n(n-1)n\, q\,\dot a\, a^{2(2-n)}\,(-K - \dot a^2 - a\,\ddot a)^{n-2}\left[\,-2\,\dot a^3 - 2K\,\dot a + a\,\dot a\,\ddot a \,+ a^2\,\dddot a\right] = 0\,,
\end{aligned} \tag{13}$$

where $\tilde\chi = \chi\rho_0 a_0^{3(1+w)}$. Let us now assume a power-law $a = a_0\,[t/t_0]^x$ for the scale factor and, for the sake of simplicity, we set $\bar a = a_0\, t_0^{-x}$ (clearly, $[\bar a] = [L^{1-x}]$). Here and in the following, we use the subscript $(...)_0$ to denote quantities measured today. In this case, Eq.(13) can be recast in the form

$$\begin{aligned}
&-6\bar a^2 K\, t^{2x} - 6\bar a^4 x^2\, t^{4x-2} + q\bar a^4\, t^{4x}\left(C_1\, t^{-2} - 6\bar a^{-2} K\, t^{-2x}\right)^n + \;2\tilde\chi\bar a^{1-3w}\, t^{x(1-3w)} \;=\\
&=\; nqx\bar a^{6-2n}\, t^{6x}\left(C_1\bar a^2\, t^{-2} - 6K\, t^{-2x}\right)^n\,\frac{(C_2 K\, t^2 + xC_3\, t^{2x})}{(K\, t^2 + C_4\, t^{2x})^2}\,,
\end{aligned} \tag{14}$$

where $C_1 = 6x(1-2x)$, $C_2 = [x(2n-1)-1]$, $C_3 = x\bar a^2(x+2n-3)(2x-1)$, and $C_4 = x\bar a^2(2x-1)$.

A. Radiation-dominated Universe

Here, we assume the radiation-dominated Universe EoS $w = 1/3$ ($\rho \sim a^{-4}$). In the following, we will separately discuss the three distinct regimes for $x < 1$, $x > 1$ and $x = 1$ in the asymptotic limit $t \to 0$,.

In the case $x < 1$, all terms containing explicitly the curvature K of Eq.(14) results to be negligible for $t \to 0$ and asymptotic solutions are allowed if and only if $x \leqslant n/2$ which, in the case $2 < n < 3$ we are considering, is always satisfied. The leading-order term of Eq.(14) writes as

$$q\bar a^4 C_1^n\,\left[\,1 - \,(C_3/C_4^2) n x^2 \bar a^2\,\right] t^{4x-2n} = 0\,, \tag{15}$$

and $x = 1/2$ and $x = [2 + 3n - 2n^2 \pm (4 + 8n + n^2 - 12n^3 + 4n^4)^{1/2}]/2n$ are the solutions. Such second expression results to be negative or imaginary for $2 < n < 3$ and must be excluded. Thus, the only solution for $x < 1$, in the asymptotic limit for $t \to 0$, is the well-known radiation dominated behavior $a \sim t^{1/2}$. In the other two cases, *i.e.*, for $x \geqslant 1$, it is easy to recognize that no asymptotic solutions are allowed. Therefore, the approach to the initial singularity is not characterized by power-law inflation behavior when spatial curvature is non-vanishing.

Let us now assume a vanishing spatial curvature in Eq.(14). In can be show how, for $K = 0$, the radiation-dominated solution with $w = 1/3$ and $x = 1/2$ is an *exact* solution (non-asymptotic and allowed for all n-values) with $\rho_0 = 3/(4\chi t_0^2)$, thus matching the standard FLRW case. In the case $x > 1$, the leading-order terms of Eq.(14) read, for $t \to 0$ and $K = 0$,

$$q\bar{a}^4 C_1^n \left[1 - (C_3/C_4^2) n x^2 \bar{a}^2\right] t^{4x-2n} + 2\tilde{\chi} = 0 \,. \tag{16}$$

Three distinct regimes have to be now separately discussed. For $x > n/2$, the leading order of the equation above does not admit solutions since it writes simply $2\tilde{\chi} = 0$ and, for $x < n/2$, the solutions of Eq.(16) are those obtained in the case for $x < 1$. Instead, for $x = n/2$, and defining $H_0 = (n/2)/t_0$, one gets

$$\rho_0 = \frac{\tilde{\rho}_0(n)\, q_0}{4\chi t_0^2}\ , \quad \tilde{\rho}_0(n) = \tfrac{3^n}{2}\,(1-n)^{(n-1)} n^n (n/2)^{2-2n}\left[4n + (6-5n)n^2 - 4\right] , \tag{17}$$

where we have introduced the dimensionless parameter $q_0 = H_0^{2n-2}\, q$. We remark that the constraint $n = (2m + 1)/(2\ell + 1)$ (which is in agreement with respect to the one obtained from Solar-System test) must hold in order to have $\rho_0 > 0$ since we have assumed $q < 0$ and therefore $q_0 < 0$. The function $\tilde{\rho}_0$ results to increase as n goes from 2 to 3 and, in particular, one can get $216 < \tilde{\rho}_0 < 21\,024$. Finally, for $x = 1$ and $K = 0$, Eq.(14) reads $[1 - n(2n - 2)]\, t^{4-2n} = 0$, giving $n = [1 \pm \sqrt{3}]/2$. As the previous case, the regime $x = 1$ does not admit solutions in the region $2 < n < 3$.

B. Matter-dominated Universe

Let us now study the matter-dominated Universe EoS $w = 0$ ($\rho \sim a^{-3}$). As previously done, we analyze the three distinct regimes for $x < 1$, $x > 1$ and $x = 1$, and, in the limit for $t \to \infty$, it is easy to recognize that there are no power-law solutions in all these cases for $K \neq 0$. Setting $K = 0$, the $x \geqslant 1$ regimes do not provide any power-law form for cosmological dynamics either. On the other hand, for $x < 1$ and assuming zero spatial curvature in Eq.(14), we get the following equation:

$$[-6x^2\bar{a}^4]\, t^{4x-2} + 2\tilde{\chi}\bar{a}\, t^x = \bar{a}^4 q C_1^n[-1 + C_3\bar{a}^2 n x^2/C_4^2]\, t^{4x-2n} \,. \tag{18}$$

Since $4x - 2 > 4x - 2n$, the term on the right hand side can be neglected in the limit of large t and the equation above admits three distinct situations: $x < 2/3$, $x > 2/3$ and $x = 2/3$. Both cases with $x \neq 2/3$ do not admit solution. The case $x = 2/3$ admits instead an asymptotic solution for $t \to \infty$. In fact, Eq.(18) reduces to $8\bar{a}^3 = 6\tilde{\chi}$ and the FLWR matter-dominated power-law solution $a = \bar{a}\, t^{2/3}$ is reached setting $\rho_0 = 4/(3\chi t_0^2)$. In conclusion, we can infer that, for $f(R) = R + qR^n$, the standard matter-dominated FLRW behavior of the scale factor $a \sim t^{2/3}$ is the only asymptotic (as $t \to \infty$) power-law solution.

As shown above, the matter dominated solution $a \sim t^{2/3}$ is obtained for $K = 0$ and asimptotically for large t. In order to neglect all the K-terms in our $f(R)$ model, we start directly from the expression of the Ricci scalar [23]. Using a power-law scale factor, we get the t-range (if $x \neq 1/2$ and $x < 1$)

$$t \ll \left|\, [x(2x-1)]/[K/\bar{a}^2]\, \right|^{1/(2-2x)} \,. \tag{19}$$

For the matter-dominated era and using standard cosmological parameters [17], one can get the upper limit $K/\bar{a}^2 \lesssim 0.006\ (H_0)^{2/3}$. Thus, setting $x = 2/3$, we get the bound $t \ll\ 235/H_0$, independently of the form of $f(R)$.

At the same time, if we set $x = 2/3$, the asymptotic solution $\rho_0 = 4/(3\chi t_0^2)$ is reached neglecting the right hand side ($\ll 1$) of Eq.(18), *i.e*, if t is constrained by the following lower limit (we remind that $q_0 = H_0^{2n-2}\, q$)

$$t \gg\ \mu(n, q_0)/H_0 \ , \qquad \mu(n, q_0) = \left|\ q_0\left[-(4/3)^n + 2^{(2n+1)}3^{-n}\, n(2n - 7/3)\right]\ \right|^{1/2(n-1)} \,. \tag{20}$$

Let us now recall that the matter-dominated era began, assuming $H_0^{-1} \simeq 4.3 \times 10^{17}$s, at $t_{Eq} \simeq 5.1 \times 10^{-6}/H_0$. In this sense, we can safely assume $\mu(n, q_0) \leqslant 5.1 \times 10^{-8}$, which implies an upper limit for $|q_0|$, *i.e.*, $|q_0| \leqslant |q_0|^{Max}$, where

$$|q_0|^{Max}(n) = \left[5.1 \times 10^{-8}\right]^{2(1-n)} \left| -(4/3)^n + 2^{(2n+1)}3^{-n}\, n(2n - 7/3)\right|^{-1} \,. \tag{21}$$

It is easy to check that the function $|q_0|^{Max}(n)$ is decreasing as n goes from 2 to 3, in particular, one gets: $10^{-16} \lesssim |q_0|^{Max} \lesssim\ 10^{-31}$.

C. Characteristic length scales of the model

As already discussed in Section 2, the parameter q has dimension $[L]^{2n-2}$. We have therefore defined a characteristic length scale of the model as $L_q(n) = |q|^{1/(2n-2)}$. Assuming $f(R)$ corrections to be smaller than the experimental uncertainty of the orbital period of the Earth around the Sun, the lower bound (10) for $L_q(n)$ was found. In order to identify the allowed scales for our model and in view of the upper constraint on the parameter $q_0 = H_0^{2n-2}\,q$ derived in the cosmological framework, we can now define the upper limit for $L_q(n)$ as

$$L_q^{Max}(n) = [|q_0|^{Max}]^{1/(2n-2)}/H_0 \,, \tag{22}$$

which, considering Eq.(21), yields to the constraints 65.59 pc $< L_q^{Max} <$ 78.37 pc, in the case $2 < n < 3$. Assuming $H_0^{-1} \simeq 4.2 \times 10^9$ pc, the two bounds for the characteristic length scales here discussed, *i.e*, Eq.(10) and Eq.(22), are plotted in Figure 1 (A). At the same time two other typical lengths have been outlined in Eq.(9) for the Solar System.

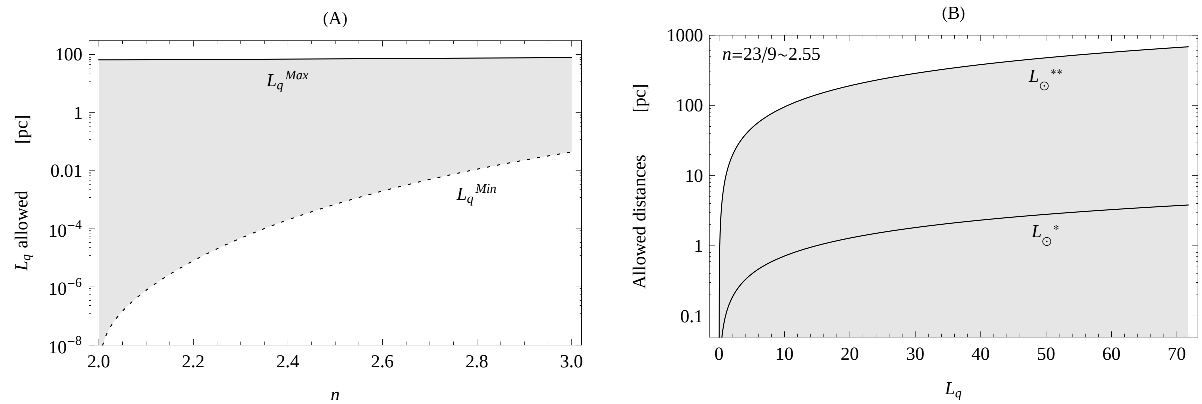

Figure 1. **Panel A:** L_q^{Min} of Eq.(10) and L_q^{Max} of Eq.(22). The gray zone represents the allowed characteristic-length scales of the model. We stress that $L_{q\odot}^{Min}$ is defined only if $n = (2m+1)/\ell$, as represented by the dotted line. **Panel B:** $L_\odot^*$ and $L_\odot^{**}$ of Eq.(9). The gray zone represents here the allowed distances for the model.

$L_\odot^*$ represents the minimum distance to have post-Newtonian and Newtonian terms of the same order, while $L_\odot^{**}$ was defined according to the request that the weak-field expansion holds. Setting now $n = 23/9 \simeq 2.55$, one can show from Eq.(10) and Eq.(22) that the allowed scales are 0.0013 pc $\lesssim L_q \lesssim$ 71.72 pc. In this range, $L_\odot^*$ and $L_\odot^{**}$ can be plotted as in Figure 1 (B).

Summarizing, our analysis states a precise range of validity for the power-law $f(R)$ model we are considering. Indeed, for a generic value of n (*i.e.*, not close to 2 or 3) the fundamental length of the model is constrained to range from the super Solar-System scale up to a sub-galactic one. Therefore, in agreement with Eq.(8a), we have to search significant modification for the Newton law in gravitational system lying in this interval of length scales, like for instance, stellar clusters (see also [24]).

D. The inflationary paradigm

After discussing the power-law evolution of the Universe proper of the radiation- and matter-dominated eras, we now analyze the inflationary behavior characterizing the very early dynamics. In this analysis, we assume that the inflationary expansion is driven by some component (*e.g.*, a scalar field) with negative pressure $p = -\rho$, as in standard inflationary models. In other words, we are interested in studying how the modification of the GR action (1) affects the de Sitter dynamics (and, indeed, if such a phase can exist in the extended framework). For an interesting approach, where the inflationary scenario is instead directly embedded within the modified gravity scheme, see [25, 26]. In this respect, we hypothesize an exponential behavior for the scale factor of the Universe $a = a_I\, e^{H_I(t-t_I)} = \bar{a}\, e^{H_I t}$, where $H_I > 0$ and $\bar{a} = a_I e^{-H_I t_I}$ (here, t_I denotes the end of inflation). In the following, we concentrate the attention on the solution for vanishing spatial curvature $K = 0$ and, in this case, Eq.(13) rewrites as

$$\bar{a}^4\, e^{4H_I t}\big[q(-12)^n\, H_I^{2n}(1-n/2) - 6H_I^2\big] + 2\tilde{\chi}(\bar{a}\, e^{H_I t})^{1-3w} = 0 \,. \tag{23}$$

Assuming now $w = -1$, *i.e.*, $\rho = \rho_I = const.$, the equation above reduces to

$$\big[(-1)^n 12^n\, q(1-n/2)\big] H_I^{2n} - 6H_I^2 + 2\chi\rho_I = 0 \,. \tag{24}$$

During inflation, ρ_I can be assumed of the order of the Grand Unification energy-scale, *i.e*, $\rho_I \simeq (10^{16}\,\mathrm{GeV})^4$. The equation above yields an algebraic relation between H_I (the Hubble constant during the de Sitter phase) and ρ_I, exactly as in the standard case; in fact, the latter is recovered for $q = 0$ and the well-known relation $H_I = \overline{H}_I = \sqrt{\chi\rho_I/3}$ holds.

In order to integrate Eq.(24), we focus on a particular value of the power-law $f(R)$ exponent, *e.g.*, $n = 29/13 \sim 2.23$. Using Eq.(10), we obtain $L_q^{Min} \simeq 1.5 \times 10^{-5}$ pc and, having in mind that $L_q = |q|^{1/(2n-2)}$ we get $|q| > |q|^{Min} \simeq 1.4 \times 10^{-12}\mathrm{pc}^{32/13}$. Let us now fix the parameter q to a reasonable value like $q^* \sim -10^3|q|^{Min}$ (we remark that such assumption will be physically motivated in the next Section). In this case, the solution of Eq.(24) is $H_I \simeq 1.8 \times 10^{22}\,\mathrm{pc}^{-1} \simeq 1.2 \times 10^{-10}\,\mathrm{GeV}$. It is worth noting that, for typical values of q like the one used above, the equation above admits solutions only in the case $n = (2\ell+1)/(2m+1)$, consistently with the previous analyses.

This study shows how an exponential early expansion of the Universe can still be associated to a constant vacuum energy, even for the modified dynamics. However, we see that the rate of expansion is significantly lower than the standard one by more than 20 orders of magnitude. In fact, the standard value is $\overline{H}_I = \sqrt{\chi\rho_I/3} \simeq 3.7 \times 10^{45}\mathrm{pc}^{-1} \simeq 2.4 \times 10^{13}\,\mathrm{GeV}$. Although our estimation relies on the assumption that ρ_I is of the order of the Grand Unification energy-scale, nevertheless the values of H_I remains many order of magnitude below the standard value even when changing ρ_I, as long as this assumes physically motivated values, as shown in Figure 2. Assuming that the minimum number of e-folds required to solve the standard model paradoxes is the usual one, this implies that the duration of the de Sitter phase should be much longer (time-wise). Despite this difference, we have shown that it is still possible to arrange the cosmological parameters in order to have a satisfactory inflationary scenario.

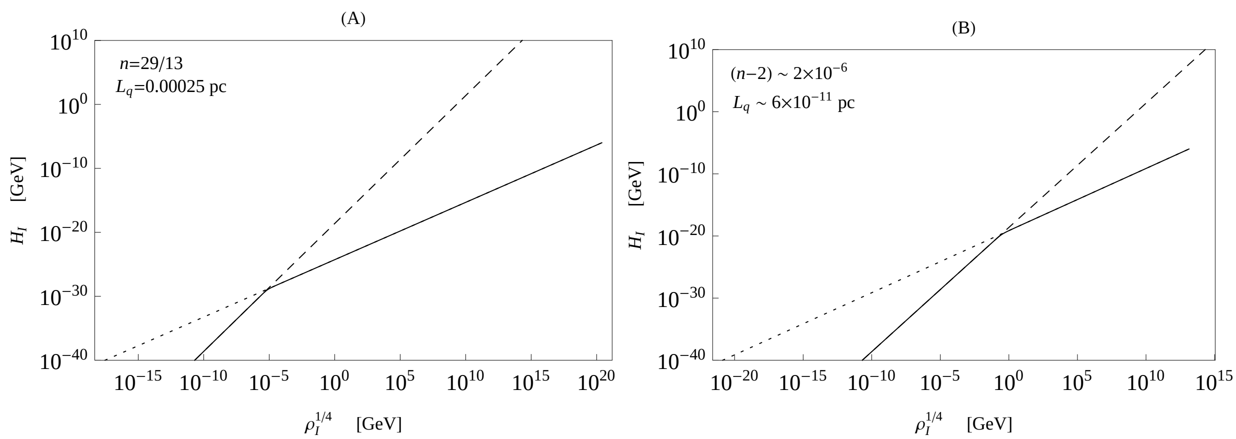

Figure 2. **Panel A**: Numerical solution of the modified Friedmann equation (24) setting $n = 29/13$ and $L_q = 2.5 \times 10^{-4}$ pc, corresponding to $q = -10^3|q|^{Min}$ (solid line). We also show the standard solution $\overline{H}_I = \sqrt{\chi\rho_I/3}$ (dashed line) and the solution obtained assuming that the $f(R)$ correction term is the leading one (dotted line), for a discussion see Eq.(27). **Panel B:** The same as the panel A, but for $(n-2) \simeq 2 \times 10^{-6}$ and $L_q \simeq 6 \times 10^{-11}$pc (again corresponding to $q = -10^3|q|^{Min}$).

Let us now analyze in more detail the implications of the modified Friedmann equation (24). First of all, we assume that the first term in this equation dominates over the second one, *i.e.*,

$$H_I \gg (n-2)^{-\frac{1}{2(n-1)}} L_q^{-1}\ , \tag{25}$$

neglecting a numerical factor of order unity. We will check *a posteriori* the parameter range for which such approximation is reasonable. With a little algebra, Eq.(24) can then be recast as

$$(-12)^{n-1}\ |q|\ (n-2)\ H_I^{2n} = \chi\rho_I/3\ . \tag{26}$$

The positiviteness of the right-hand side requires that $(-1)^{n-1} = 1$ and then that n is of the form $(2\ell+1)/(2m+1)$, as anticipated above. In this case, the relation (26) can be explictly inverted to give:

$$H_I = \left[\frac{4\chi\rho_I}{12^n|q|(n-2)}\right]^{\frac{1}{2n}} = \left[\frac{4\chi\rho_I}{12^n(n-2)}\right]^{\frac{1}{2n}} \times \left(\frac{1}{L_q}\right)^{\frac{n-1}{n}}\ . \tag{27}$$

Denoting with $m_q \equiv L_q^{-1}$ the energy scale associated to the parameter q and with $M = \rho_I^{1/4}$ the energy scale of

inflation, and recalling that $\chi = 8\pi/m_{\rm pl}^2$ we can write (neglecting a numerical factor of order unity):

$$H_I \sim \left[\frac{M^2 m_q^{n-1}}{m_{\rm pl}\sqrt{n-2}}\right]^{\frac{1}{n}} . \tag{28}$$

This should be compared with the standard value $\overline{H}_I = \sqrt{\chi\rho_I/3} \sim M^2/m_{\rm pl}$. The suppression factor for the expansion velocity thus roughly amounts to

$$\frac{H_I}{\overline{H}_I} \sim \frac{1}{(n-2)^{\frac{1}{2n}}}\left(\frac{m_{\rm pl}m_q}{M^2}\right)^{\frac{n-1}{n}} . \tag{29}$$

For consistency, we should check that the solution (27) satisfies the condition (25). This is the case if

$$\rho_I = M^4 \gg \frac{1}{32\pi(n-2)^{\frac{1}{n-1}}}\frac{m_{\rm pl}^2}{L_q^2} . \tag{30}$$

We are going to argue that this is practically always the case, due to the existence of the lower bound (10). In fact, since $L_q \geqslant L_q^{Min}(n)$, one can argue that, if the above condition is satisfied for $L_q = L_q^{Min}(n)$, then it is automatically satisfied for all the values of L_q allowed by the Solar-Systems tests. We have said that this is *practically* always the case, because the right-hand side of Eq.(30) blows up when $n \to 2$, thus in this limit it is increasingly difficult to satisfy the condition on the energy density expressed by this equation. This is related to the fact that the $f(R)$ correction to the Friedmann equation (24) during inflation vanishes for $n = 2$. However, we will show in a moment that, for reasonable values of ρ_I, this is not a problem unless the value of $(n-2)$ is unnaturally tuned to be very small. When n is not too close to 2 (for definiteness, let us take $(n-2) \gtrsim 10^{-2}$) the right hand side of Eq.(30), calculated at $L_q = L_q^{Min}(n)$, lies roughly between $(10^{-7}\,\mathrm{GeV})^4$, for $n = 3$, and $\sim (10^{-3}\,\mathrm{GeV})^4$, for $n = 2.01$. The energy scale of inflation is poorly constrained by observations; however, inflation certainly has to involve some physics beyond the standard model, and we can safely take $M \gtrsim 1\,\mathrm{TeV}$. By doing this, we see that the condition (30) is always satisfied as long as n is not very close to 2. More quantitatively, we find that $(n-2) > 2\times 10^{-3}$ is enough to ensure that $M > 10\,\mathrm{MeV}$. Although $(n-2)$ could be very small in principle, nevertheless having $(n-2) \simeq 10^{-3}$ or smaller seems like an undesirable fine tuning of the model. It can be easily understood that the amount of fine tuning required in order to spoil the condition (30) gets larger when considering higher values of the inflation energy scale. The above reasoning is confirmed by the inspection of Figure 3, where we show the allowed values of M, the energy scale of inflation, as a function of n. Finally, we also note that the condition (30) implies, through Eq.(29), that in this regime $H_I/\overline{H}_I \ll 1$.

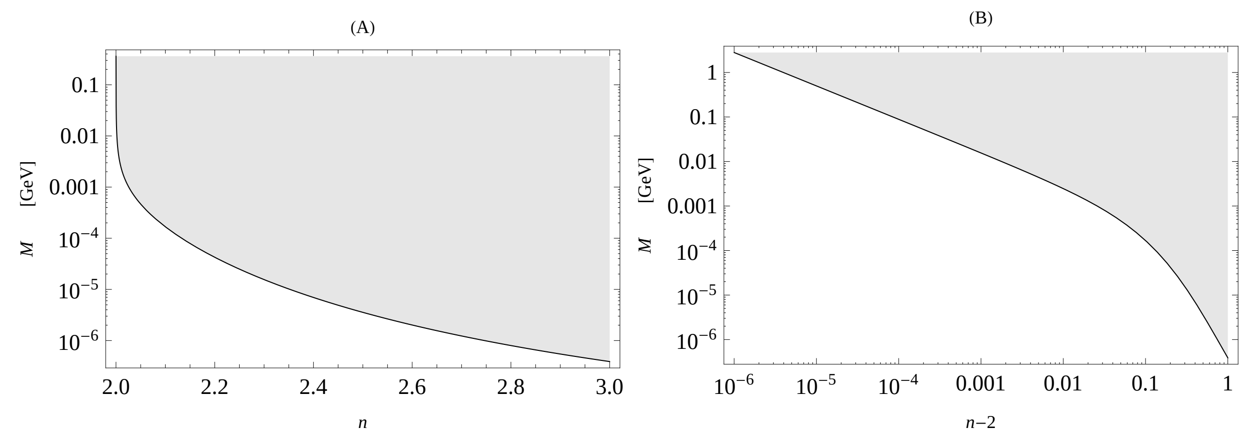

Figure 3. **Panel A:** In grey: region in the $M-n$ plane where the $f(R)$-term in the modified Friedmann equation dominates during inflation, irregardless of the value of L_q (as long as Solar-System bounds are satisfied). We remark that this term could dominate also in the region below the curve, depending on L_q. **Panel B**: The same as panel A, but this time in the $M-\log(n-2)$ plane, in order to highlight the behaviour for $n \to 2$.

We conclude this section by discussing an important observational implication of the smallness of H_I during inflation with respect to its GR value $\overline{H}_I$. It is well known that tensor perturbations h_{ij} to the background metric are produced during inflation. These perturbations physically correspond to GW fluctuations and give origin to a cosmological

background that could be, in principle, detected through measurements of the CMB temperature and polarization spectra. The amplitude of the tensor perturbations is expressed through their power spectrum $P_T(k)$, quantifying the variance of fluctuations with wave number k. In standard GR, this is found to be

$$P_T(k) = 4\chi \left(\frac{H}{2\pi}\right)^2 \Bigg|_{k=aH} . \tag{31}$$

where the k dependence comes from the fact that the right-hand side is evaluated at the time the mode of interest leaves the horizon ($k = aH$). Since H is nearly constant during inflation, the value of H at horizon crossing is, in a first approximation, the same for all k-modes; this implies that the spectrum should be nearly scale-invariant. In the following we will neglect the scale dependence of the spectrum and always take $H = H_I$. It has been shown [27] that, in the framework of modified theories of gravity, the amplitude of the spectrum is modified as follows:

$$P_T = \frac{4\chi}{|f'(R)|} \left(\frac{H_I}{2\pi}\right)^2 . \tag{32}$$

The $f'(R)$ term is given by:

$$f'(R) = 1 + nqR^{n-1} = 1 + n\left(\frac{4\chi\,\rho_I\,L_q^2}{n-2}\right)^{\frac{n-1}{n}} , \tag{33}$$

where we have used the fact that during the de Sitter phase $R = 12H_I^2$, and we have substituted expression (27) for H_I. It is straightforward to verify that the condition (30) ensures that the second term in the equation above is always much larger than unity; thus we have

$$f'(R) \simeq n\left(\frac{4\chi\,\rho_I\,L_q^2}{n-2}\right)^{\frac{n-1}{n}} = n\left[\frac{32\pi\,M^4}{(n-2)m_q^2 m_{\rm pl}^2}\right]^{\frac{n-1}{n}} \gg 1 . \tag{34}$$

This implies that the amplitude of the primordial spectrum of tensor modes is strongly suppressed, first because of the large factor $f'(R)$ appearing in Eq.(32), secondly because the value of H_I is itself suppressed with respect to its standard value, as discussed above. Substituting the expressions (27) and (34) for f' and H_I into Eq.(32) yields:

$$P_T = \frac{\chi}{12n\pi^2\,L_q^2}\left[\frac{n-2}{4\chi\rho_I L_q^2}\right]^{\frac{n-2}{n}} = \frac{2}{3n\pi}\left(\frac{m_q^2}{m_{\rm pl}^2}\right)\left[\frac{(n-2)m_{\rm pl}^2 m_q^2}{32\pi M^4}\right]^{\frac{n-2}{n}} . \tag{35}$$

It is interesting to note that, contrarily to the usual, the amplitude of tensor modes is smaller for large values of M, and that in the limit $n \to 2$, it does not depend on M but only on m_q, *i.e.*, $P_T|_{n\to 2} \simeq (1/6\pi)(m_q^2/m_{\rm pl}^2)$. It is useful to write the tensor spectrum in terms of the prediction of standard GR, given by Eq.(31) evaluted for $H_I = \overline{H}_I$, times a factor β, that will turn out to be very small. By doing this, we obtain

$$P_T = \frac{(n-2)}{n}^{\frac{n-2}{n}}\left(\frac{m_{\rm pl}^2 m_q^2}{32\pi M^4}\right)^{\frac{2(n-1)}{n}} \overline{P}_T \equiv \beta\overline{P}_T , \tag{36}$$

where $\beta = \beta(n,\, m_q,\, M)$. The amplitude of the spectrum is clearly maximized in the case $m_q = 1/L_q^{Min}(n)$, *i.e.*, when the Solar-System bounds are saturated. As discussed above, this maximum allowed mass is a decreasing function of n, formally diverging for $n \to 2$, but still cannot be made very large unless n is fine-tuned to be very close to 2. For $n = 2.01$, we get the tiny value $L_q^{Min} = (5 \times 10^{-25}\,{\rm GeV})^{-1}$, and it is easy to check that, in order for the factor $m_{\rm pl}m_q/M^2$, that roughly controls the magnitude of β, to be of order unity, one would need M to lie at or below the MeV scale, that is completely unrealistic. This is made clearer in Fig. 4, where we show curves of constant β in the (M, n) and $(M, \log(n-2))$ planes. We can conclude that reasonable values of n and M yield $P_T \ll \overline{P}_T$.

The amplitude of tensor modes is measured by the ratio r of tensor perturbations to scalar (density) perturbations, evaluated at the pivot wavenumber $k_0 = 0.002\,{\rm Mpc}^{-1}$, *i.e.*, $r \equiv (P_T/P_S)|_{k=k_0}$. The primordial amplitude of scalar perturbations is well measured by CMB experiments to be $P_S|_{k=k_0} \simeq 2.4 \times 10^{-9}$; using this, Eq.(35) can be readily

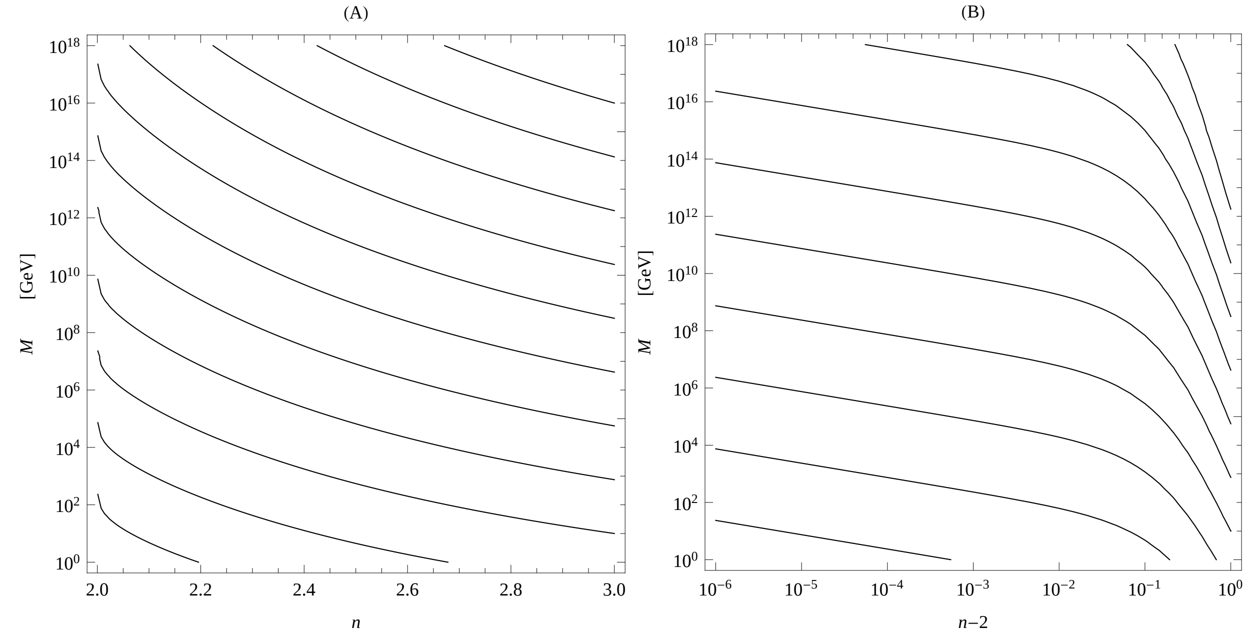

Figure 4. **Panel A:** Curves of constant β in the $(M,\, n)$ plane. The value of β decreases going from left to right and from bottom to top. From left to right, the plotted curves correspond to $\beta = \{10^{-20},\, 10^{-30},\, \ldots,\, 10^{-120}\}$. In all cases m_q is assumed to saturate the Solar-System bounds, in order to give the largest possible value for β. **Panel B:** The same as panel A, but in the $(M,\, \log(n-2))$ plane. From left to right, the plotted curves correspond to $\beta = \{10^{-10},\, 10^{-20},\, \ldots,\, 10^{-100}\}$.

turned into an expression for r. The tensor-to-scalar ratio is constrained by the recent WMAP data to be $r < 0.36$ at 95% confidence level (c.l.); inclusion of other datasets lowers the bound down to $r < 0.20$ at 95% c.l. [28, 29]. The present constraints are essentially driven by the temperature and temperature-polarization cross-correlation data [30]. The incoming data from the *Planck* satellite will likely improve the 95% c.l. bound to $r \lesssim 0.04$ [31]. Tighter constraints could be obtained through better measurements (meaning either a detection or a strong upper bound) of the so-called B-modes of the CMB polarization: in principle, a future space-based mission focused on CMB polarization, like the CMBPol mission currently under study, could detect tensor modes at 99% c.l. for $r \gtrsim 0.01$ *or* provide an upper limit $r \lesssim 0.002$ at the same c.l. [32]. However, expression (35) yields a value of $r \sim 10^{-80}$ for $n = 2.01$ (practically independent from M), and still lower values for larger n (see Figure 5). This implies that the extended theory of gravity considered here predicts a cosmological background of gravitational waves that will be beyond experimental reach even in the far future, at least for the very large wavelengths probed by CMB experiments. This is made even more clear by the plots in Figure 6, where it is evident that "large" (*i.e.*, potentially detectable) values of r are only possible in a tiny region of parameter space corresponding to unnaturally small values of $n-2$. More precisely, since in the limit of small $n-2$ the asymptotic behaviour of r is $r \simeq 10^{-83}(n-2)^{-2}$ (independently of M, and optimistically assuming that $L_q = L_q^{Min}$), it is straightforward to see that, in order to have a tensor-to-scalar ratio at a level of 10^{-4} or larger, $n-2$ has to be of the order of 10^{-40} or smaller.

The smallness of the tensor-to-scalar ratio can be considered as a well-definite, falsifiable prediction of the model. The immediate implication of the discussion above is that a detection of a non-zero value of the tensor-to-scalar ratio in near future would rule out the model considered here, unless one is willing to admit that n is fine-tuned to be incredibly close to 2, but yet, not exactly equal to 2, as this value is not admitted by the theory.

V. PROPAGATION OF GRAVITATIONAL WAVES ON A RW BACKGROUND

The design and the construction of a number of sensitive detectors for GWs is underway today. There are some laser interferometers like the VIRGO detector, built in Cascina (near Pisa, Italy) by a joint Italian-French collaboration, the GEO 600 detector built in Hanover (Germany), by a joint Anglo-German collaboration, the two LIGO detectors built in the United States one in Hanford (Washington) and the other in Livingston (Louisiana) by a joint Caltech-MIT

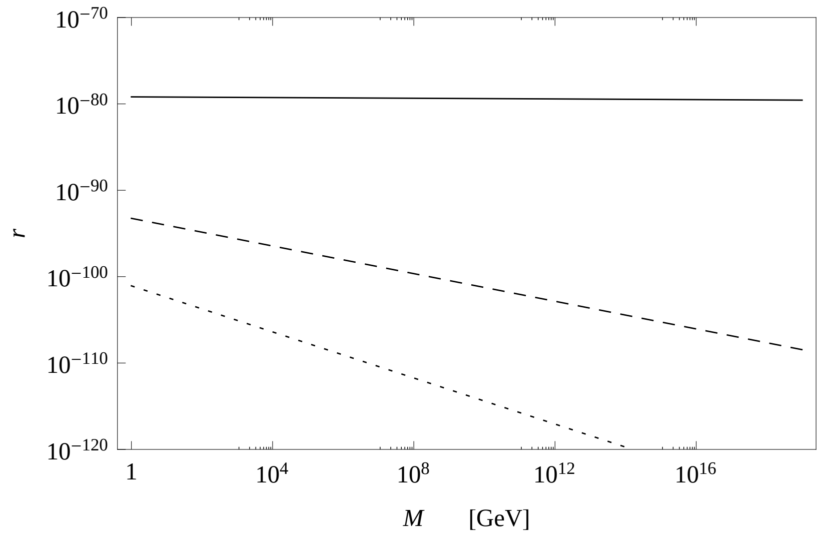

Figure 5. Tensor-to-scalar ratio r as a function of the inflationary energy scale M, for different values of the index n. Solid line $n = 2.01$; dashed line $n = 2.5$; dotted line $n = 3$. In all cases m_q is assumed to saturate the Solar-System bounds, in order to give the largest possible value for r.

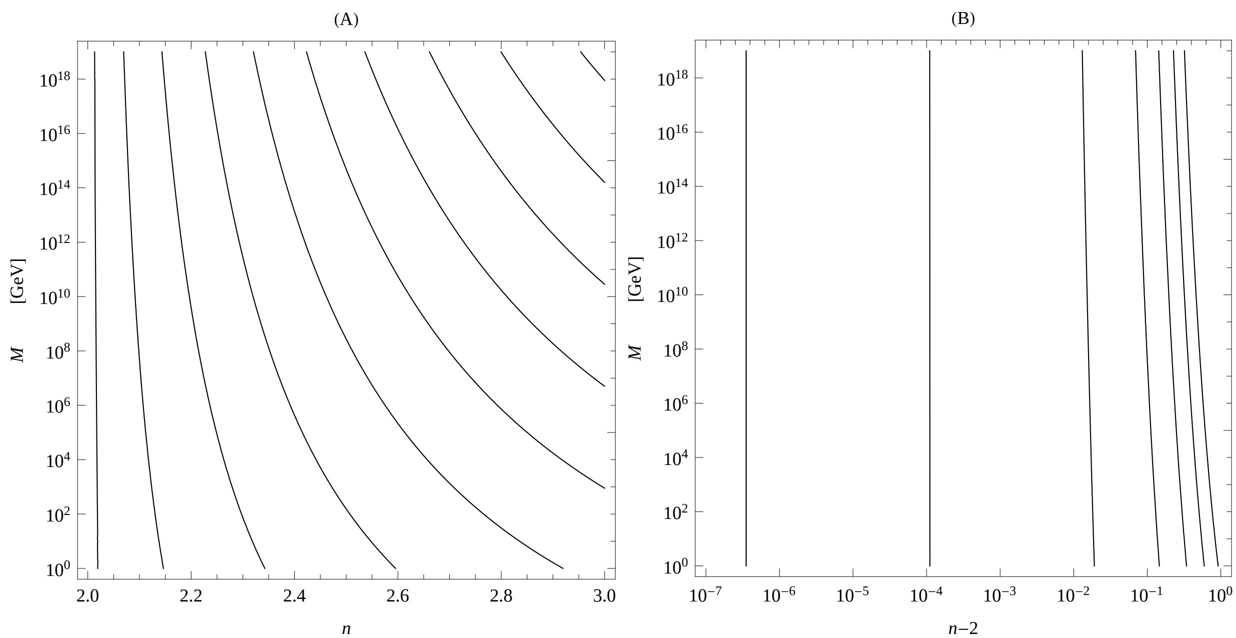

Figure 6. **Panel A:** Curves of constant r in the (M, n) plane. The value of r decreases going from left to right and from bottom to top. From left to right, the plotted curves correspond to $r = \{10^{-80}, 10^{-85}, \ldots, 10^{-125}\}$. In all cases m_q is assumed to saturate the Solar-System bounds, in order to give the largest possible value for r. **Panel B:** The same as panel A, but in the $(M, \log(n-2))$ plane. From left to right, the plotted curves correspond to $r = \{10^{-70}, 10^{-75}, \ldots, 10^{-100}\}$.

collaboration, and the TAMA 300 detector, in Tokyo (Japan). Many bar detectors are currently in operation too, and several interferometers and bars are in a phase of planning and proposal stages (for the current status of GW experiments see [33]). The results of these detectors will have a fundamental impact on astrophysics and gravitation physics. There will be a huge amount of experimental data to be analyzed, and theorists would face new aspects of physics from such a data stream. Furthermore, GW detectors will be of fundamental importance to probe the GR or every alternative theory of gravitation [34]. A possible target of these experiments is the so-called stochastic background of GWs. The production of the primordial part of this stochastic background (relic GWs) is well known

in literature starting from [35], in which, using the so-called adiabatically-amplified zero-point fluctuations process, it has been shown in two different ways how the inflationary scenario for the early universe can, in principle, provide the signature for the spectrum of relic GWs.

The GW equations in the transverse-traceless gauge write $\Box h_i^j = 0$ and they are derived from the Einstein field equations deduced from the EH Lagrangian $\mathcal{L} = R$. Clearly, matter perturbations do not appear in the GW equations since scalar and vector perturbations do not couple with tensor perturbations in the Einstein equations [2]. For a general derivation of the propagation equation of GWs in the extended framework, we start considering Eq.(1) without specifying, in the first instance, the form of $f(R)$. For the sake of simplicity, we discard matter contributions and a conformal analysis will help to this goal. In fact, assuming the conformal transformation

$$\tilde{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu} \qquad \text{with} \qquad e^{2\phi} = f'(R) \,, \tag{37}$$

where ϕ is the *conformal scalar field*, we obtain the conformally equivalent HE action

$$\mathcal{S} = \tfrac{1}{2\chi} \int d^4x \sqrt{-g} \left[\tilde{R} + \mathcal{L}\left(\phi,\, \phi_{;\,\mu}\right)\right] \,. \tag{38}$$

Here $\mathcal{L}\left(\phi,\, \phi_{;\,\mu}\right)$ is the conformal scalar field contribution derived from

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + 2(\phi_{;\mu}\phi_{;\nu} - g_{\mu\nu}\phi_{;\delta}\phi^{;\delta} - \phi_{;\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\phi^{;\delta}{}_{;\delta}) \,, \quad \tilde{R} = e^{-2\phi}(R - 6\Box\phi - 6\phi_{;\delta}\phi^{;\delta}) \,. \tag{39}$$

Starting from the action (38) and deriving the Einstein-like conformal equations, expressed in the conformal metric $\tilde{g}_{\mu\nu}$, the GW equations writes as

$$\tilde{\Box}\, \tilde{h}_i^j = 0 \,. \tag{40}$$

Since no scalar perturbation couples to the tensor part of GWs, one can easily recognize that the relation $\tilde{h}_i^j = \tilde{g}^{lj}\delta\tilde{g}_{il} = e^{-2\phi} g^{lj} e^{2\phi}\delta g_{il} = h_i^j$ holds, which means that h_i^j is a conformal invariant. As a consequence, the plane-wave amplitude

$$h_i^j = h(t)\, \epsilon_i^j\, e^{ik_i x^i} \,, \tag{41}$$

where ϵ_i^j is the polarization tensor, are the same in both metrics. In any case, the d'Alembert operator transforms as

$$\tilde{\Box} = e^{-2\phi}\left(\Box + 2\phi^{;\lambda}\nabla_\lambda\right) \,, \tag{42}$$

and this means that the background is changing while the tensor wave amplitude not.

In order to study the cosmological stochastic background, the operator (42) can be specified for a RW flat model, *i.e.*, $K = 0$ in Eq.(11), and then the propagation equation (40) becomes

$$\ddot{h} + [3H + 2\dot{\phi}]\,\dot{h} + k^2 a^{-2}\, h = 0 \,, \tag{43}$$

being $\Box = \partial_t^2 + 3H\partial_t$ and $k = k_i k^i$ the comoving wave number. It is worth stressing that Eq.(43) applies to any $f(R)$ theory whose conformal transformation can be defined as $e^{2\phi} = f'(R)$ and, thus, the solution, *i.e.*, the GW amplitude, depends on the specific cosmological background (*i.e.*, $a(t)$) and on the specific theory of gravity (*i.e.*, $\phi(t)$).

We start by studying a non-analytical $f(R)$ model with a power-law scale factor. We assume a power-law behavior for the scale factor, *i.e.*, $a = a_0\,[t/t_0]^x$ (we restrict ourselves to the case $x < 1$), and the non-analytical form of $f(R)$ given in Eq.(1). Using the relation $\phi(t) = (1/2)\ln f'$ and $R_{RW} = 6x(2x-1)/t^2$ (for $K = 0$), we get

$$\phi(t) = \frac{1}{2}\ln\left[1 + nqR^{n-1}\right] = \frac{1}{2}\ln\left[1 + nq\big(6x(2x-1)\big)^{n-1} t^{-2(n-1)}\right] \,. \tag{44}$$

It is straightforward to see that during the radiation-dominated era ($x = 1/2$), $\phi(t)$ vanishes identically. Thus Eq.(43) reduces to its GR form and the standard result for the propagation of GWs follows, *i.e.*, $h \propto \sin(k\eta)/k\eta$, where η is the conformal time, defined by the relation $dt = ad\eta$.

For $x \neq 1/2$, the time derivative of the field ϕ is given by:

$$\dot{\phi} = \frac{n(1-n)q\big[6x(2x-1)\big]^{n-1} t^{1-2n}}{1 + nq\big[6x(2x-1)\big]^{n-1} t^{-2(n-1)}} = \frac{(n-1)}{t}\frac{1}{(t/t_*)^{2(n-1)} - 1} \,, \tag{45}$$

where we defined $t_* = (n|q|)^{1/2(n-1)} \, |6x(2x-1)|^{\frac{1}{2}}$. We remark that the second equality is valid for n in the form $(2\ell+1)/(2m+1)$, as required by the model. Thus, it is easy to see that for $t \ll t_*$, $\dot{\phi} \simeq (1-n)/t$. Since $H \simeq 1/t$, in this regime $\dot{\phi}$ and H are comparable and we expect the $f(R)$ action to affect the propagation of GWs. When $t \sim t_*$, $\dot{\phi}$ becomes very large (and eventually diverges for $t = t_*$) and the corrections to the EH action dominate the dynamics. Finally, when $t \gg t_*$, we have that $|\dot{\phi}| \ll H$ and the standard behavior is recovered.

Let us now briefly discuss the GW propagation in the regime $t \ll t_*$, when $\dot{\phi} = (1-n)/t$. In this case, Eq.(43) can be recast in the form

$$\ddot{h} + \left[3x + 2(1-n)\right] \frac{\dot{h}}{t} + \frac{k^2}{a^2} h = 0 \, . \tag{46}$$

It is convenient to switch to the time variable $u \equiv k\eta$. Eq. (46) thus rewrites as

$$\frac{d^2 h}{du^2} + \frac{2(x-n+1)}{(1-x)u} \frac{dh}{du} + h = 0 \, . \tag{47}$$

The main advantage of this variable choice is that the explicit dependence on the wave number k disappears from the dynamical equation. Furthermore, since η represents the cosmological horizon, it is easily understood that $u = 1$ roughly corresponds to the time of horizon crossing, and that the regime $u \ll 1$ ($u \gg 1$) represents a wave that is far outside (inside) the horizon. The general solution to this equation is

$$h(u) = u^\alpha \left[A_1 J_\alpha(u) + A_2 Y_\alpha(u)\right] \, , \qquad \alpha = \frac{2n-3x-1}{2(1-x)} \, , \tag{48}$$

where J_α and Y_α are the Bessel functions of the first and second kind, respectively, and the integration constants A_1, A_2 are fixed by the initial conditions. A useful insight can be gained analyzing the asymptotic behavior of the above solution for $u \ll 1$ and $u \gg 1$. In the range of interest ($2 < n < 3$ and $0 < x < 1$), we have that $\alpha > 3/2$ and we can safely use the following expansions for the Bessel functions:

$$J_\alpha(u) \approx \frac{1}{\Gamma(\alpha+1)} \left(\frac{u}{2}\right)^\alpha \, , \qquad Y_\alpha(u) \approx -\frac{\Gamma(\alpha)}{\pi} \left(\frac{u}{2}\right)^{-\alpha} \, , \qquad u \ll 1 \, ; \tag{49a}$$

$$J_\alpha(u) \approx \sqrt{\frac{2}{\pi u}} \cos\left(u - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \, , \quad Y_\alpha(u) \approx \sqrt{\frac{2}{\pi u}} \sin\left(u - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \, , \quad u \gg 1 \, . \tag{49b}$$

Thus the two independent solutions in Eq.(48) are both well-behaved for $u \to 0$; in particular, the solution $u^\alpha J_\alpha(u) \to 0$, while $u^\alpha Y_\alpha(u)$ tends to a constant value. The first derivatives of both solutions are also vanishing in $u = 0$, as it can be shown by using the relations between the derivative of J_α in terms of $J_{\alpha-1}$ and $J_{\alpha+1}$ (and similarly for Y_α). On the other hand, the behavior of the solution when the wave is well inside the horizon ($u \gg 1$) shows some interesting features. In fact, in this regime both solutions oscillate with an amplitude that *grows* with time like $u^{\alpha-1/2}$ (we recall that $\alpha > 3/2$); this is in striking contrast with the usual behavior that is found in standard GR, where the amplitude decays with time (*e.g.*, $h \propto \sin(u)/u$ during the radiation-dominated era). This is related to the fact that the coefficient of the first-derivative term in Eq.(47), that is positive in standard GR, becomes negative for $n > x+1$, or, equivalently, for $\alpha > 1/2$.

As already said, during the radiation-dominated era, $\dot{\phi}$ vanishes identically and the analysis above does not apply. In principle, however, the propagation of GWs during the matter-dominated era ($x = 2/3$) could be modified with respect to the standard picture, depending on the value of t_*, that we are now going to estimate. As discussed in the previous section, the value of $|q|$, or equivalently of L_q, is bounded from above by the requirement of the existence of the matter-dominated era itself. The exact value of L_q^{Max} depends on n, but only weakly, so that we can safely consider that $L_q \lesssim 70$ pc. Then we find that $t_* \lesssim 400\,$yr; since the matter-dominated era began at $t \simeq 10^5\,$yr, we can conclude that during this epoch the contribution of the $\dot{\phi}$ term is negligible and thus the propagation of GWs is standard.

To conclude, we briefly turn our attention to the GW propagation during a de Sitter phase. We then hypothesize an exponential behavior for the scale factor of the Universe $a = \bar{a}\, e^{H_I t}$, with $H_I > 0$, as done in Section IV. In this case, using the relation $\phi(t) = (1/2) \ln f'$, one can argue that

$$\phi = \tfrac{1}{2} \, \ln(nq) + \tfrac{1}{2} \, (n-1) \, \ln\left[12 H_I^2 + 6K e^{-2H_I t}/\bar{a}^2\right] \, , \tag{50}$$

and, of course, if we focus on the case of vanishing spatial curvature $K = 0$, one gets $\dot{\phi} = 0$. Since, as it can be seen from Eq.(43), the modifications to the GR action only enter the GW dynamics through a term proportional to $\dot{\phi} = 0$, this implies that their propagation during a de Sitter phase is the same as in standard GR.

VI. CONCLUDING REMARKS

In this work, we have considered a modified theory of gravity of the form $f(R) = R + qR^n$, where both $q < 0$ and $2 < n < 3$ are free parameters.

The present analysis has demonstrated how, in contrast with analogous treatments [8], the non-analytic case in exam is not excluded by Solar-System tests. Instead, using the measurements of the Earth period, a lower bound for the characteristic length scale $L_q = |q|^{1/(2n-2)}$ of the proposed model can be found, shedding light on the viability of the theory in the weak-field limit. Such constraint confines non-Newtonian effects to be relevant outside the Solar System only, unless n is fine-tuned to be very close to 2. We underline that our proposal for a suitable value of the characteristic length could induce, in principle, relevant modifications in larger-scale astrophysical systems, *e.g.*, stellar clusters.

The model has also been implemented in a cosmological framework, showing the possibility to recover the main phases of the Universe evolution, *i.e.*, the radiation- and matter-dominated era. We have assumed a power-law behavior of the cosmological scale factor and have shown how these two phases are reproduced in the relevant limits. In fact, the radiation-dominated evolution is recovered (irregardless of the spatial curvature) in the asymptotic limit toward the singularity. In the same limit, a phase of power-law inflation was also found for vanishing curvature. On the other hand, in the limit of large times, the matter-dominated phase is reproduced in the case of zero curvature. As a main result, an upper bound of ~ 70 pc for the characteristic length was found by guaranteeing the existence of such an era of the evolution of the Universe. The presence of this cosmological bound and of the Solar-System constraints, has allowed us to define a restricted region of the parameter space in which the model can be implemented both in the weak-field limit and in the cosmological framework.

Moreover, we have shown how an exponential early evolution of the Universe is still present in the extended $f(R)$ scheme. The expansion rate was found to be much smaller than the standard value by several orders of magnitude due to the presence of a minimum length scale for the model. This allows to have the inflationary scale arbitrarily close to the Planck one without spoiling the current limits on the tensor-to-scalar ratio r, since the latter is proportional to the expansion rate and thus gets correspondingly suppressed. More generally, we have found that a tiny value of the tensor-to-scalar ratio is expected in the extended framework unless the exponent n is fine tuned to be unnaturally close to 2 (but still different from 2). The observational implication is that, if the gravitational action is of the form considered here, tensor modes will not be detected even by next-generation CMB-polarization experiments. On the other hand, a detection of tensor modes in the near future would falsify the present model.

The issue of the GW propagation on a RW background has also been addressed, assuming a power-law as well as an exponential behavior of the cosmological scale factor. We have shown the presence of a typical time scale over which $f(R)$ corrections can be relevant, which implies that the term due to the generalized framework can always be neglected during the matter-dominated era. On the other hand, the corrective term vanishes identically both in the radiation-dominated era and in the de Sitter phase. This way, the GW evolution results to be, for all practical purposes, the same as in standard GR. Combined with the suppression of primordial tensor modes, this implies that the inflationary GW background will be below the detection threshold of present and future interferometers, and thus represents another observational test for the model. On the contrary, different production mechanisms (*e.g.*, cosmic strings) could still, in principle, yield a cosmological GW signal of sizable intensity.

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